Core Course –06
Semestar - III
Paper : Group Theory –I
Lesson: Binary Composition

Lesson Developer : Sukhendu Roy
Department/ College : Department of Mathematics ,
Mankar College, Burdwan University

Binary Composition

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1. **Introduction:** In Mathematics , a binary operation on a set G is simply a calculation which combines two elements of the set (called operands) to produce another element of the set. In other words a binary operation on a set is simply a method or formula by which the members of an ordered pair from a set(Say G) combine to yield a new member of G. The most familiar binary operations are ordinary addition, subtraction, multiplication of integers.

2. **Objectives:** After studying this chapter you should be able to :
   - ❖ Define binary operation
   - ❖ State the laws of operations
   - ❖ Cayley(Composition) Table
   - ❖ Solve the related problems for different examinations

3. **Binary Operation:**

Defination: Let A be a non empty Set and A x A ={(a,b) : a $\in A, b \in A\}$. A binary compostion(or operation) on A is a mapping f: A$\times A \rightarrow A$. Therefore a binary operation f assigns a definite element of A to each ordered pair of elements of A & this mapping is generally denoted by the symbol $'o'$.

The symbols like * , $\odot$, $\oplus$, **.** are also used to denote a binary operation.

- **Important Points to remember :**

   I. If a Set A contains n elements , then the number of binary operations on A are n^{n^2}

   II. Division of integers is not a binary operation on the integers since an integer divided by an integer need not be an integer.

   III. If a & b are any two elements of A then image of (a,b) under the binary operation o denoted by aob . If a,b,c be three elements of A then the image of (a,(b,c)) under binary operation o is denoted by ao (boc)

   IV. Let A be a nonempty Set. A binary operation ‘ o ‘ on A is said to be defined if a o b ∈ A ∀ a, b ∈ A. In this Case the set A is said to be closed under (or closed with respect to) the binary composition ‘ o ‘

4. Examples of Binary Operation :

- On the Set ℤ , let o stands for the binary operation ‘+’(addition) then 3 o 2 = 5 for 3,2 ∈ ℤ
- On the Set ℝ , let o stand for the binary operation ‘ Subtraction’ then 2 o 3 = - 1 for 2,3 ∈ ℝ
- Let a binary operation * be defined on the set ℚ by a*b = $\frac{ab}{3}$ then 3*5 = 5 , 3*8 =24 for 3 ,5 ,8 ∈ ℚ
- The Set ℕ is not closed under ‘ subtraction’ , since for some a ,b ∈ ℕ , a-b does not belongs to ℕ $always$ e.g 2 ,3 ∈ ℕ but 2-3 = - 1∉ ℕ

5. Laws Of Binary Operations:

5.1 Commutative Property:
Let A be a non empty set .A binary operation o on A is said to be commutative if aob = boa ∀ a, b ∈ A.

Examples:

1) If o stand for binary operation ‘- ‘ on ℕ then aob≠ boa ∀ a, b ∈ ℕ
2) If o stand for binary operation ‘+ ‘ on ℝ then aob = boa ∀ a, b ∈ ℝ
3) If o stand for binary operation ‘ **.** ‘ on ℝ then aob = boa ∀ a, b ∈ ℝ

5.2 Associative Property :
Let A be a non emty set.

A binary operation o on A is said to be associative if
ao(boc)= (aob)oc $\forall a, b, c \in A$

Examples:

1) If o stands for the binary operation '+' on $\mathbb{R}$ then ao(boc) = (aob)oc $\forall a, b, c \in \mathbb{R}$
2) If o stands for the binary operation ' - ' on $\mathbb{N}$ then ao(boc) $\neq (aob)oc \ \forall a, b, c \in \mathbb{N}$
3) If o stands for binary operation ' . ' then it is associative over $\mathbb{Z}$, $\mathbb{Q}$ & C

5.3 Distributive Property :

Let A be a non empty set. Let o & * be two binary operations on A. Now the binary operation o on A is said to be left distributive over * if ao(b*c)= (aob)*(aoc) $\forall a, b, c \in A$ & right distributive over * if (b*c) o a = (boa) *(coa) $\forall a, b, c \in A$

Examples:

- If o stands for binary operation ' . ' and * stands for binary operation ' + ' on $\mathbb{R}$ then binary operation o is distributive over *
- If o stands for binary operation ' + ' and * stands for binary operation ' **.** ' on $\mathbb{R}$ then binary operation o is not distributive over * for $\mathbb{R}$

5.4 Cancellation Laws :

Let A be a non empty set. A binary operation o on A is said to satisfy the cancellation law given below:

- Left Cancellation Law : a o b = a o c $\Longrightarrow b = c$ $\forall a, b, c \in A$
- Right Cancellation Law : b o a = c o a $\forall a, b, c \in A$

Examples :

a) If o stand for binary operation ' **.** ' on $\mathbb{N}$ then it satisfy the left & right cancellation law
b) If o stand for binary operation ' **.** ' on $\mathbb{R}$ then it does not satisfy the left & right cancellation law over $\mathbb{R}$. Also if o stand for the binary operation ' **.** ' then it does not satisfy the cancellation law over $\mathbb{Z}$, $\mathbb{Q}$ & C

5.5 Composition (Cayley) Table :

A cayley table ,after 19th century British mathematician Arthur Cayley describes the structure of a finite group by arranging all the possible products of all the group's elements in a square table.

Let A be a non empty finite set then a binary composition o on the set A can be defined in tabular form which is called cayley table (or composition table)

Let G = { $a_1, a_2, a_3, \dots \dots \dots . . a_n$} be the set of n elements.These n elements are to be listed across the top & at the left of table in same order.Here $a_i o\ a_j$ appears on the table in i-th row & j-th column as given in the table below.

o	a_1	a_2	a_3	....	a_n
a_1	$a_1 o a_1$	$a_1 o a_2$	$a_1 o a_3$	....	$a_1 o a_n$
a_2	$a_2 o a_1$	$a_2 o a_2$	$a_2 o a_3$	....	$a_2 o a_n$
a_3	$a_3 o a_1$	$a_3 o a_2$	$a_3 o a_3$	....	$a_3 o a_n$
⋮	⋮	⋮	⋮	⋮	⋮
a_n	$a_n o a_1$	$a_n o a_2$	$a_n o a_3$	....	$a_n o a_n$

➢ **Important Points to remember :**

- ❖ If the Cayley table is symmetric about its principal Diagonal then binary operation o is called commutative.
- ❖ If the set A contains n elements, then the elements are arranged in n^2 entries into n-rows & n columns .They are also elements of A since A is closed under binary operation o .
- ❖ The binary operation addition modulo n & multiplication modulo n on the set {0,1,2,3,........n-1} which we denote by $\mathbb{Z}_n$ play an important role in abstract algebra.

5.6 Examples of Cayley Table :

I. The table for the binary composition ' addition modulo 3 ' on the set $\mathbb{Z}_3$ is given as follows :

$\mathbb{Z}_3$ = the set of all residue classes of modulo 3 = { $\bar{0}$, $\bar{1}$, $\bar{2}$ }

$+_3$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

Here the Composition Table shows that the Composition is commutative.

II. Let G = { 1 , ω , ω^2} be the set of all cube roots of unity and the multiplication as a binary operation defined on G. Here all the elements are arranged in $3^2 = 9$ entries into 3 rows & 3 columns and they are also the elements of G. Now the composition table for the binary operation ' multiplication ' on the Set G is given as follows :

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Here the above multiplication table is symmetric about its principal diagonal.
So the binary operation multiplication is commutative on G.

III. The table for the binary composition ' multiplication modulo 5 ' on the set $\mathbb{Z}_5$ is given as follows :

$\mathbb{Z}_5 =$ the set of all residue classes of modulo 5
= { $\bar{0}$, $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$}

$\times_5$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

6. Miscellaneous Problems :

Problem 1 :

A binary operation o is defined on $\mathbb{R}$, the set of real numbers by a o b = a+ 2b $\forall\ a, b \in \mathbb{R}$. Is the binary operation o is commutative & associative ? Justify

Solution :

Step 1 : To examine commutativity let us take a ,b $\in$ $\mathbb{R}$ then aob = a+2b and boa = b+2a. Therefore ao b $\neq boa$
e.g let us take 1, 2 $\in$ $\mathbb{R}$. Here 1 o 2 = 1+ 2$\times$ 2 $= 5$ for 1 , 2$\in$ $\mathbb{R}$

and 2 o1 =2 + 2$\times$ 1 = 4 for 1 , 2 $\in$ $\mathbb{R}$. Therefore 1o2 $\neq 2o1$
Hence 'o' is not commutative on $\mathbb{R}$.

Step 2 : To examine associativity let us take a,b, c $\in$ $\mathbb{R}$. Then ao(boc) = a o (b+2c) = a + 2 $\times$ $(2 + 2c) = a + 4c + 4$ and (aob) o c = (a+2b) o c = a+2b+2c .
Therefore a o (b o c) $\neq (aob)oc$
e.g let us take 2 ,4, 5 $\in$ $\mathbb{R}$. Then 2 o (4 o 5) = 2 o (4+ 2$\times$ 5) $= 2$ o 14

= 2 + 2 $\times$ 14 $= 30$. Now (2 o 4) o 5 = (2 + 2$\times$ $4)o5 = 10\ o\ 5 = 10 + 2 \times 5 = 20$.
Hence 2 o (4 o 5) $\neq$ (2 o 4) o 5.
Therefore ' o ' is also not associative on $\mathbb{R}$.

Problem 2 :

A binary operation o is defined on $\mathbb{R}$, the set of real numbers by a o b = |ab| $\forall\ a, b \in$ $\mathbb{R}$. Is the binary operation o is commutative & associative ? Justify

Solution : Here the binary operation o is defined on $\mathbb{R}$ by
a o b = |ab| $\forall\ a, b \in$ $\mathbb{R}$

Step 1 : To examine the commutativity let us take $a, b \in$ $\mathbb{R}$.
Here a o b = |ab| $\forall\ a, b \in$ $\mathbb{R}$ & b o a = |ba| $\forall\ a, b \in$ $\mathbb{R}$. i.e a o b = b o a $\forall\ a, b \in$ $\mathbb{R}$
Therefore ' o ' is commutative on $\mathbb{R}$

Step 2 : To examine associativity let us take a , b, c $\in$ $\mathbb{R}$
Here a o (b o c) = a o |bc| = |a |bc| | = |abc| $\forall\ a, b, c \in$ $\mathbb{R}$
Again (a o b) o c = |ab| o c = | |ab| c| = |abc| $\forall\ a, b, c \in$ $\mathbb{R}$. Therefore a o (b o c) $= (aob)oc$ $\forall\ a, b, c \in$ $\mathbb{R}$
Hence ' o ' is also associative on $\mathbb{R}$

Problem 3 :

A binary operation o is defined on $\mathbb{Z}$, the set of integers by a o b = a+b +1 $\forall\ a, b \in \mathbb{Z}$. Is the binary operation o is commutative & associative ? Justify

Solution : Here the binary operation o is defined on $\mathbb{Z}$ by
a o b = a + b +1 $\forall\ a, b \in$ $\mathbb{Z}$

Step 1 : To examine the commutativity let us take $a, b \in$ $\mathbb{Z}$
Here a o b =a + b +1 $\forall\ a, b \in$ $\mathbb{Z}$ and b o a = b + a + 1 = a +b +1 i.e a o b = b o a $\forall\ a, b \in$ $\mathbb{Z}$. Therefore ' o ' is Commutative on $\mathbb{Z}$.

Step 2 : To examine associativity let us take a , b, c ∈ $\mathbb{Z}$.
Then a o (b o c) = a o (b +c +1) = a + (b + c+ 1) + 1 = a + b +c +2 $\forall a, b, c \in \mathbb{Z}$.
Again (a o b) o c = (a + b +1) o c = (a + b + 1) + c +1 = a + b + c + 2 $\forall a, b, c \in \mathbb{Z}$. Hence ' o ' is also associative on $\mathbb{Z}$.

Problem 4 :

A binary operation o is defined on $\mathbb{Z} \times \mathbb{Z}$ by (a , b) o (c , d) = (a- c , b -d) ; $\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$. Is the binary operation o is commutative & associative ? Justify.

Solution : Here the binary operation o is defined on $\mathbb{Z} \times \mathbb{Z}$ by
(a , b) o (c , d) = (a- c , b -d) ; $\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$.

Step 1 : To examine the commutativity let us take $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$. Then by defination (a , b) o (c , d) = (a- c , b- d) & (c , d) o (a, b) = (c – a , d- b) = (- (a-c) , - (b-d))
Therefore (a , b) o (c , d) ≠ (c , d) o (a, b) $\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$.
This shows that ' o ' is not commutative on $\mathbb{Z} \times \mathbb{Z}$

Step 2 : To examine associativity let us take $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z}$.
Then (a , b) o { (c , d) o (e ,f) } = (a,b) o {(c-e ,d-f)} [by defination]
= (a – c +e , b – d+ f) [by defination]
Again { (a,b) o (c,d)} o (e,f) = (a – c , b – d) o (e ,f)
= (a – c – e , b – d – f)
Hence (a , b) o { (c , d) o (e ,f) } ≠ { (a,b) o (c,d)} o (e,f) $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z}$. Therefore ' o ' is also not associative on $\mathbb{Z} \times \mathbb{Z}$.

Problem 5 : A binary operation o is defined on $\mathbb{Q}$, the set of rational no. by a o b = ab+1 $\forall a, b \in \mathbb{Q}$. Is the binary operation o is commutative & associative ? Justify.

Solution : Here the binary operation o is defined on $\mathbb{Q}$ by
a o b = ab +1 $\forall a, b \in \mathbb{Q}$

Step 1 : To examine the commutativity let us take $a, b \in \mathbb{Q}$.
Then a o b = ab +1 = ba +1 [since ab = ba $\forall a, b \in \mathbb{Q}$]
= b o a
Thus a o b = b o a $\forall a, b \in \mathbb{Q}$. Hence ' o ' is commutative on $\mathbb{Q}$.

Step 2 : To examine associativity let us take a , b, c ∈ $\mathbb{Q}$.
Then a o (b o c) = a o (bc +1) = a (bc +1) +1 = abc + a +1 ∀ a , b, c ∈ $\mathbb{Q}$.
Again we have (a o b) o c = (ab + 1) o c = (ab + 1)c + 1 = abc + c + 1

$\forall$ a , b, c $\in \mathbb{Q}$. Hence a o (b o c) $\neq$ (a o b) o c $\forall$ a , b, c $\in \mathbb{Q}$.
Therefore it proves that ' o ' is not associative on $\mathbb{Q}$.

Problem 6 :

A binary operation o is defined on $M_2(\mathbb{R})$, the set of all 2× 2 matrices whose elements are real numbers by A o B = $\frac{1}{2}(AB - BA)\ \forall A, B \in M_2(\mathbb{R})$. Is the binary operation o is commutative & associative ? Justify .

Solution : Here the binary operation o is defined on $M_2(\mathbb{R})$ by
A o B = $\frac{1}{2}(AB - BA)\ \forall A, B \in M_2(\mathbb{R})$.

Step 1 : To examine the commutativity let us take A , B $\in M_2(\mathbb{R})$

Then A o B = $\frac{1}{2}(AB - BA)\ \forall A, B \in M_2(\mathbb{R})$ and

B o A = $\frac{1}{2}(BA - AB) = -\frac{1}{2}(AB - BA)\ \forall A, B \in M_2(\mathbb{R})$

Therefore A o B $\neq B\ o\ A\ \forall A, B \in M_2(\mathbb{R})$.

Hence ' o ' is not commutative on $M_2(\mathbb{R})$.

Step 2 : To examine associativity let us take A , B , C $\in M_2(\mathbb{R})$. Then

A o (B o C) = A o $\frac{1}{2}$(BC – CB) = $\frac{1}{2}\ [\frac{1}{2}A(BC - CB) - \frac{1}{2}$(BC- CB)A]

$= \frac{1}{4}\ [A(BC - CB) - (BC - CB)A]$

$= \frac{1}{4}\ [ABC - ACB - BCA + CBA]$

Again (A o B) o C = $\frac{1}{2}(AB - BA)o\ C = \frac{1}{2}\ [\frac{1}{2}(AB - BA)C - \frac{1}{2}C(AB - BA)]$

$= \frac{1}{4}\ [(AB - BA)C - C(AB - BA)]$

$= \frac{1}{4}\ [ABC - BAC - CAB + CBA\ ]$

$\neq \frac{1}{4}\ [ABC - ACB - BCA + CBA]$;

[since matrix multiplication is not commutative in general]

Therefore A o (B o C) $\neq$ (A o B) o C $\forall A, B, C \in M_2(\mathbb{R})$.

Hence 'o ' is not associative on $M_2(\mathbb{R})$.

- **The following table will give Comprehensive ideas :**

SL.NO	The Given Binary Operation	Nature of Binary operation	
		Commutativity	Associativity
1	A binary operation o is defined on $\mathbb{R}$ by a o b = a+ 2b $\forall a, b \in \mathbb{R}$	NO	NO
2	A binary operation o is defined on $\mathbb{R}$ by a o b = |ab| $\forall a, b \in \mathbb{R}$.	Yes	Yes
3	A binary operation o is defined on $\mathbb{Z}$ by a o b = a+b +1 $\forall a, b \in \mathbb{Z}$.	Yes	Yes

4	A binary operation o is defined on $\mathbb{Z} \times \mathbb{Z}$ by (a , b) o (c , d) = (a- c , b -d) $\forall\ (a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}$.	No	NO
5	A binary operation o is defined on $\mathbb{Q}$ by a o b = ab+1 $\forall\ a,b \in \mathbb{Q}$	Yes	NO
6	A binary operation o is defined on $M_2(\mathbb{R})$ by A o B = $\frac{1}{2}(AB - BA)\ \forall\ A, B \in\ M_2(\mathbb{R})$	NO	NO

Problem 7 :

Let ' o ' be an associative binary Composition on a Set S. Let T be a subset of S defined by T = { a ∈ $S : aox = xoa\ \ \forall\ x \in S$ } . Prove that T is closed under ' o ' .

Solution : Here by given Condition c we have

$$a\,o\,x = x\,o\,a \ \ \&\ \ b\,o\,x = x\,o\,b\ \ \forall\ x \in S$$

Now we have to Prove that T is closed under ' o ' i.e for a , b $\in T$

$$\Longrightarrow aob \in T$$

Now $(aob)o\,x = a\,o\,(\,b\,o\,x)$ [since o is associative]

$\Longrightarrow\ (aob)o\,x = a\,o\,(x\,o\,b)$ [since b o x = x o b $\forall\ x \in S$]

$\Longrightarrow\ (aob)o\,x = (a\,o\,x)o\,b$ [since o is associative]

$\Longrightarrow\ (aob)o\,x = (x\,o\,a\,)o\,b$ [since a o x = x o a $\forall\ x \in S$]

$\Longrightarrow\ (aob)o\,x = x\,o\,(a\,o\,b)$ [since o is associative]

Thus we have $(aob)o\,x = x\,o\,(a\,o\,b)\ \ \forall\ x \in S$. This implies that $aob \in T\ \ aob \in T$ .Therefore T is closed under 'o' .

Problem 8 :

Let S be a Set of two elements . How many different binary Compositions can be defined on S ? How many different Commutative binary compositions can be defined on S ?

Solution : Here it is given that S be a set of two elements. Now we know that the number of different binary compositions defined on S be n^{n^2} where n denotes number of elements.

Therefore the total no. Of different binary compositions defined On the Set S = $2^{2^2} = 2^4 = 16$ [Since here n = 2]

❖ The total number of different commutative binary Compositions defined on the set S be $n^{\frac{n(n+1)}{2}} = 2^{\frac{2(2+1)}{2}} = 2^3 = 8$ [Since here n = 2]

Learning Outcomes :

- The associativity & the Commutativity are independent of each other.
- If G contains n elements then the number of binary operations on G are n^{n^2}
- If a set G contains n elements , then the number of commutative binary operations on G are $n^{\frac{n^2+n}{2}}$
- If a set G contains n elements , then the number of non-commutative binary operations on G are $n^{n^2} - n^{\frac{n^2+n}{2}}$.

Exercise :

1. Prove or disprove : Every binary operation on a Set S is both commutative & associative iff S has exactly one element.
2. Examine the composition 'o' defined on the given set are i) commutative ii) associative .

   (a) ' o' on $\mathbb{R}$ defined by ao b = |a| + |b| $\forall\, a, b \in \mathbb{R}$.
   (b) ' o' on $\mathbb{N}$, the set of all natural numbers by a o b = L.C.M of a & b $\forall\, a, b \in \mathbb{N}$
   (c) ' o' on $\mathbb{N}$, the set of all natural numbers by a o b = gcd (a , b) $\forall\, a, b \in \mathbb{N}$
   (d) ' o' on $\mathbb{N}$, the set of all natural numbers by a o b = max (a , b) $\forall\, a, b \in \mathbb{N}$
   (e) ' o' on $\mathbb{Z}$, the set of all integers by a o b = a + b – ab $\forall\, a, b \in \mathbb{Z}$

3. Prove or disprove : Every Commutative binary operation on a set G containing two elements is associative.
4. Let S be a set of 5 elements . How many different commutative & non-commutative binary compositions can be defined on the set S ?

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